



TITLE:

The chordal norm of Kleinian groups(Analysis of Discrete Groups II)

AUTHOR(S):

Furusawa, Harushi

CITATION:

Furusawa, Harushi. The chordal norm of Kleinian groups(Analysis of Discrete Groups II). 数理解析研究所講究録 1997, 1022: 71-76

ISSUE DATE:

1997-12

URL:

<http://hdl.handle.net/2433/61706>

RIGHT:

The chordal norm of Kleinian groups

Harushi Furusawa(Kanazawa Gakuin Univ.)

Let $M\ddot{o}b$ denote the group of all orientation preserving mobius transformations of the extended complex plane $\hat{C} = C \cup \{\infty\}$. We associate with each

$$f(z) = \frac{az + b}{cz + d} \in M\ddot{o}b, ad - bc = 1,$$

the martix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C)$$

and set $tr(f) = tr(A)$, where $tr(A)$ denote the trace of A . Note that $tr(f)$ is defined up to sign. The matrix norm $m(f)$ is defined by

$$m(f) = \|A - A^{-1}\| = (2|a - d|^2 + 4|b|^2 + 4|c|^2)^{\frac{1}{2}}.$$

The quantity $m(f)$ is independent of the choice of A representing f .

For each f and g in $M\ddot{o}b$ we let $[f, g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4, \gamma(f, g) = tr([f, g]) - 2,$$

the parameters of the two generator group $\langle f, g \rangle$. These parameters are independent of the choice of representative matrices for f and g , and they determine $\langle f, g \rangle$ up to conjugacy whenever $\gamma(f, g) \neq 0$. We derive a lower bound for the distance in the metric of

$$(1) \quad d(f, g) = \sup\{q(f(z), g(z)); z \in \hat{C}\}$$

where $\langle f, g \rangle$ is a Kleinian group generated by f, g in $M\ddot{o}b$ and q denotes the chordal distance in \hat{C} ,

$$q^2(z, w) = \frac{4|z - w|^2}{(|z|^2 + 1)(|w|^2 + 1)}.$$

A mobius transformation h is said to be a chordal isometry if

$$q(h(z), h(w)) = q(z, w)$$

for all $z, w \in \hat{C}$, then $m(f)$ is invariant with respect to conjugation by chordal isometries.

If $f \in M\ddot{o}b$ with $fix(f) = \{z_1, z_2\}$, then

Lemma 1. Let f be loxodromic or elliptic with $fix(f) = \{z_1, z_2\}$ then

$$(2) \quad 2|\beta(f)| = \frac{q(z_1, z_2)^2}{8 - q(z_1, z_2)^2} m(f)^2.$$

Proof. All quantities in (2) are invariant with respect to conjugation by chordal isometries. Therefore by means of such a conjugation we may arrange that $z_1 = -r$ and $z_2 = r$ where $0 < r \leq 1$. Then f is represented by

$$A = \begin{pmatrix} a & br \\ br^{-1} & a \end{pmatrix}, b^2 = a^2 - 1.$$

Hence $\beta(f) = 4b^2, q(-r, r) = \frac{4r}{r^2+1}$, and $m(f)^2 = \|A - A^{-1}\|^2 = (r^2 + r^{-2})|2b|^2$. Thus we have

$$m(f)^2 = 2 \frac{8 - q(-r, r)^2}{q(-r, r)^2} |\beta(f)|.$$

Lemma 2. Suppose that f is in $Möb \setminus \{id\}$ with $d(f, id) < 2$. If f has two fixed points and if 2θ is the argument of its multiplier, then

$$(3) \quad m(f)^2 \leq 8 \cos^2 \theta \frac{d(f, id)^2}{4 - d(f, id)^2}$$

with equality if and only if f has antipodal fixed points.

Lemma 3. If f is elliptic of order n , then

$$(4) \quad d(f, id) \geq 2 \sin\left(\frac{\pi}{n}\right)$$

with equality if and only if $n = 2$ or f is primitive with antipodal fixed points.

Proof. If f has order n and $fix(f) = \{-r, r\}$, then $m(f)^2 = 4(r^2 + r^{-2}) \sin^2 \theta$ where $\theta = k\pi/n$ and $1 \leq k < n$. Thus if $d(f, id) < 2$, we have

$$d(f, id)^2 \geq \frac{16(r^2 + r^{-2}) \sin^2 \theta}{8 \cos^2 \theta + 4(r^2 + r^{-2}) \sin^2 \theta} \geq 4 \sin^2 \theta$$

from Lemma 2, and the right hand side of above inequality is an increasing in $(r^2 + r^{-2})$. We derive $d(f, id)^2 \geq 4 \sin^2 \theta \geq 4 \sin^2 \pi/n$ with equality when $n > 2$ if and only if $r = k = 1$.

Lemma 4. If f and g are in $Möb$, then

$$(5) \quad \max\{d(f, id), d(g, id)\} \geq \left\{ \frac{4|\gamma(f, g)|^{\frac{1}{2}}}{2 + |\gamma(f, g)|^{\frac{1}{2}}} \right\}^{\frac{1}{2}}$$

Proof. If A and B are in $SL(2, C)$, then $\|AB - BA\| \leq \frac{1}{8} \|A - A^{-1}\|^2 \|B - B^{-1}\|^2$. Suppose that f and g are represented by A and B in $SL(2, C)$, then we have $|\gamma(f, g)| =$

$|tr([A, B]) - 2| = |det((AB)(BA)^{-1} - I)| = |det(AB - BA)| \leq \frac{1}{2} \|AB - BA\|^2 \leq \frac{1}{16} \|A - A^{-1}\|^2 \|B - B^{-1}\|^2 = \frac{1}{16} m(f)^2 m(g)^2$. It is easily seen that

$$m(f)^2 \leq \frac{8d(f, id)^2}{4 - d(f, id)^2},$$

from Lemma 2 where equality holds if f is either the identity or hyperbolic with antipodal fixed points. Without loss of generality, set $m(g) \leq m(f)$, then we have

$$4|\gamma(f, g)|^{\frac{1}{2}} \leq m(f)m(g) \leq m(f)^2 \leq \frac{8d^2}{4 - d^2}.$$

Therefore we have the result.

Lemma 5 Suppose that f and g are in $M\ddot{o}b$ and that $f(H) = g(H) = H$. If f and g are hyperbolic with $\gamma(f, g) < 0$, then there exists h in $M\ddot{o}b$ such that $h(H) = H$ and $2\beta(f_1) = m(f_1)^2, 2\beta(g_1) = m(g_1)^2$, where $f_1 = hfh^{-1}, g_1 = hgh^{-1}$ and each of f_1, g_1 have antipodal fixed points.

Theorem 6 Suppose that $\langle f, g \rangle$ is a Kleinian subgroup of $M\ddot{o}b$ and that f and g have no common fixed point and are not both of order 2. Then

$$(6) \quad \max\{d(f, g), d(f^{-1}, g^{-1})\} \geq k_1$$

where k_1 is an absolute constant, $0.853 \leq k_1 \leq 0.911\dots$

Proof. Let $\gamma = \gamma(f, g) = \gamma(fg^{-1}, g^{-1})$ and $\beta = \beta(fg^{-1}) = \beta(g^{-1}f)$. If fg^{-1} is of order n and where $n = 2, 3, 4$, or 6 , then $d(f, g) = d(fg^{-1}, id) \geq 2\sin(\pi/n)$ from Lemma 3. If $\gamma(fg^{-1}, g^{-1}) = \beta(fg^{-1})$, then [M2] implies that fg^{-1} is elliptic of order $2, 3, 4$, or 6 or g is elliptic of order 2 . Therefore we assume that $\gamma \neq \beta$, and consider the subgroup $\langle fg^{-1}, g^{-1}f \rangle = \langle fg^{-1}, g^{-1}(fg^{-1})g \rangle$ of $\langle fg^{-1}, g^{-1} \rangle$ with $\gamma(fg^{-1}, g^{-1}f) = \gamma(fg^{-1}, g^{-1})\{\gamma(fg^{-1}, g^{-1}) - \beta(fg^{-1})\} = \gamma(\gamma - \beta) \neq 0$. Thus we have

$$(7) \quad |\gamma(fg^{-1}, g^{-1}f)| \geq 2 - 2\cos(\pi/7)$$

Therefore we have

$$\begin{aligned} \max\{d(f, g), d(f^{-1}, g^{-1})\} &= \max\{d(fg^{-1}, id), d(g^{-1}f, id)\} \\ &\geq \left(\frac{4(2\cos(\frac{\pi}{7}) - 2\cos(\frac{\pi}{7}) + 1)}{2\cos(\frac{\pi}{7}) - 2\cos(\frac{\pi}{7}) + 3} \right)^{\frac{1}{2}} \geq 0.853. \end{aligned}$$

From now on, we show an upper bound for d , let $\langle \phi, \psi \rangle$ denote the $(2, 3, 7)$ triangle group acting on the upper half plane H with $\phi^2 = \psi^3 = (\phi\psi)^7 = id$ and set $f = \phi\psi$ and $g = \psi\phi$. Then $\gamma = tr([\phi, \psi]) - 2 = 2\cos(\frac{2\pi}{7}) - 1$, $\beta = tr^2([\phi, \psi]) - 4 = 2(\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1) > 0$ where $fg^{-1} = [\phi, \psi]$. Hence fg^{-1} and $g^{-1}f$ are hyperbolic with $\gamma(fg^{-1}, g^{-1}f) = \gamma(\gamma - \beta) < 0$. Then Lemma 5 show that there exists a mobius transformation h of H such that

$\text{fix}(hfg^{-1}h^{-1}) = \{z_1, z_2\}$ and $\text{fix}(hg^{-1}fh^{-1}) = \{w_1, w_2\}$ where $q(z_1, z_2) = q(w_1, w_2) = 2$. Then $2\beta(hfg^{-1}h^{-1}) = m(hfg^{-1}h^{-1})^2$, $2\beta(hg^{-1}fh^{-1}) = m(hg^{-1}fh^{-1})^2$ and $hfg^{-1}h^{-1}$ and $hg^{-1}fh^{-1}$ are both hyperbolic with antipodal fixed points. Therefore we have the result

$$\begin{aligned} d(f, g) &= d(fg^{-1}, id) = 2 \left(\frac{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1}{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) + 1} \right)^{\frac{1}{2}} \\ d(f^{-1}, g^{-1}) &= d(g^{-1}f, id) = 2 \left(\frac{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1}{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) + 1} \right)^{\frac{1}{2}} \end{aligned}$$

from Lemma 2 and hence that $k_1 \leq 0.911$.

Theorem 7 Suppose that f and g are elements of a Kleinian subgroup G of $Möb$ which are not both of order 2, 3, 4, or 6. Then f and g commute or

$$(8) \quad \max\{d(f, g), d(f^{-1}, g^{-1})\} \geq k_1$$

Proof. It suffices to consider the case where $\gamma(f, g) = 0$ and $fg \neq gf$, then

$$\text{fix}(f) \cap \text{fix}(g) \neq \phi, \text{fix}(f) \neq \text{fix}(g)$$

Then $\langle f, g \rangle$ is elementary and $h = [f, g]$ is parabolic. We complete the proof by showing that fg^{-1} is elliptic of order $n (= 2, 3, 4 \text{ or } 6)$ and $d(f, g) = d(fg^{-1}, id) \geq 1$.

Suppose that $\text{fix}(h) = \{\infty\}$. If f or g , say f , is parabolic, then g is elliptic of order 2, 3, 4 or 6 and an elementary calculation shows that the same is true of fg^{-1} . If f and g are both elliptic, then

$$f(z) = \nu z + a, g(z) = \mu z + a, fg^{-1}(z) = \frac{\nu}{\mu}z + c$$

where $\nu^p = \mu^q = (\frac{\nu}{\mu})^r = 1$, $p, q \in \{2, 3, 4, 6\}$ and $r \in \{1, 2, 3, 4, 6\}$. By hypothesis $p \neq q$, $\nu \neq \mu$ and fg^{-1} is of order 2, 3, 4, 6.

Example. If $f(z) = \lambda z$ and $g(z) = \lambda z - c$ where $\lambda^p = 1, 0 < |c| \leq 2$ and $p = 2, 3, 4 \text{ or } 6$, then $\langle f, g \rangle$ is discrete while

$$d(f, g) = d(f^{-1}, g^{-1}) = \frac{8|c|}{4 + |c|^2} \rightarrow 0$$

as $c \rightarrow 0$. It is necessary to make the hypothesis that f and g are not both of order 2, 3, 4 or 6 in Theorem 7.

Remark. For each $1 < b < a < \infty$ let $f = f_0g_0$ and $g = g_0$ where

$$f_0(z) = a^2z, g_0(z) = \frac{(b^2 + 1)z + 2b}{2bz + (b^2 + 1)}$$

Then $\langle f, g \rangle$ is nonelementary Kleinian while

$$(9) \quad d(f, g) = d(f_0, id) = 2 \left(\frac{a^2 - 1}{a^2 + 1} \right) \rightarrow 0$$

as $a \rightarrow 1$. Hence there exists no universal lower bound for $d(f, g)$ and $d(f^{-1}, g^{-1})$.

These theorems give a geometric estimate of how different two mobius transformations must be in order to generate a nonelementary Kleinian group.

Theorem 8 Suppose that $\langle f, g \rangle$ is a Kleinian group and f and g have no common fixed point and are not both of order 2. If fg is also not of order 2, then

$$(10) \quad \max\{d(fg, gf), d((fg)^{-1}, (gf)^{-1})\} \geq k_1.$$

Proof. Suppose that g is not of order 2 and let $\gamma = \gamma(f, g)$ and $\beta = \beta(fg)$. If $\gamma = \beta$, then $\beta([f, g]) = \gamma(\gamma + 4) = -3, -4$ and thus that $[f, g]$ is elliptic of order 2 or 3. Hence

$$d(fg, gf) = d([f, g], id) \geq \sqrt{3}.$$

Otherwise $\langle fg, gf \rangle = \langle fg, g(fg)g^{-1} \rangle$ is Kleinian with

$$\gamma(fg, gf) = \gamma(fg, g)\{\gamma(fg, g) - \beta(fg)\} = \gamma(\gamma - \beta) \neq 0,$$

then $k_1 \geq 0.853$ from Theorem 6. Next let $\langle f, g \rangle$ be the group which (6) holds with equality in Theorem 6. Then $f = \phi\psi$ and $g = \psi\phi$ where $\langle \phi, \psi \rangle$ is the triangle group with $\phi^2 = \psi^3 = (\phi\psi)^7 = id$ and we obtain

$$d(f, g) = d(f^{-1}, g^{-1}) = 0.911.$$

from Theorem 5. Hence the group $\langle \phi, \psi \rangle$ shows that $k_1 \leq 0.911$.

Gehring and Martin showed that if $\langle f, g \rangle$ is a nonelementary Kleinian subgroup of $M\ddot{o}b$, then

$$(11) \quad m(f)m(g) \geq 4(\sqrt{2} - 1) = 1.656\dots,$$

follows from Jørgensen's inequality and the proof of Lemma 4. In the proof of Lemma 4, we have $16|\gamma(f, g)| \leq m(f)^2 m(g)^2$ and if $\langle f, g \rangle$ is a nonelementary Kleinian group then, $m(f)m(g) \geq 4\sqrt{|\gamma(f, g)|} \geq 1.780$.

The following result shows that the average of the chordal norms of the generators f and g is always bounded below by a constant k_1 and $d(g, id) \rightarrow 2$ as $d(f, id) \rightarrow 0$ uniformly in the collection of all nonelementary Kleinian groups $\langle f, g \rangle$.

Theorem 9 Suppose that $\langle f, g \rangle$ is nonelementary Kleinian group of $M\ddot{o}b$. Then

$$(12) \quad d(f, id) + d(g, id) \geq 2k_1, 2d(f, id) + d(g, id) \geq 2$$

Proof. We may assume that $2a = d(f, id) + d(g, id) \leq 2$ in the proof for both parts of (12) and that $d(g, id) \leq d(f, id)$ in the proof for the first inequality in (12). Next above assumption together with the first inequality in (12) imply $2 < \frac{3}{2}\{d(f, id) + d(g, id)\} \leq 2d(f, id) + d(g, id)$ whenever $d(f, id) \geq d(g, id)$, hence we may also assume that $d(f, id) \leq d(g, id)$ in the proof for the second inequality in (12). Then $d(f, id) = a - x$ and $d(g, id) = a + x$ where $0 \leq x < a$ and we obtain

$$16|\gamma(f, g)| \leq m(f)^2 m(g)^2 \leq \frac{8d(f, id)^2}{4 - d(f, id)^2} \frac{8d(g, id)^2}{4 - d(g, id)^2}$$

from Lemma 4. Let $\phi(x) = \{4(a-x)^{-2}-1\}\{4(a+x)^{-2}-1\}$ and $\phi(x) \leq 4\{2-2\cos(\pi/7)\}^{-1}$ by Cao. Since $\phi(x)$ is increasing with respect to $[0, a]$ where $0 < a \leq 1$ and we have

$$a \geq 2 \left(\frac{\sqrt{|\gamma(f, g)|}}{2 + \sqrt{|\gamma(f, g)|}} \right)^{\frac{1}{2}}$$

This establishes the first part of (12) with $k_1 \geq 0.853...$

If $2 - d(g, id) > 2d(f, id)$, then $a + x < 2\{1 - (a - x)\}$ and

$$\psi(y) = (4y^{-2} - 1)\{(1 - y)^{-2} - 1\} < \phi(x)$$

where $y = a - x$. By elementary calculus,

$$\begin{aligned} \psi'(y) &= -8y^{-2}(1-y)^{-3}(5y-2)(y-2) \\ \psi(y) &\geq \psi(2/5) > 40 > 4\{2-2\cos(\pi/7)\}^{-1} \end{aligned}$$

for $0 < y < 1$ and we have a contradiction. This establishes the second part of (12).

References

- [B] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, 1983.
- [C1] C. Cao, Some trace inequalities for discrete groups of mobius transformations, Proc. Amer. Math. Soc. 123(1995).
- [C2] C. Cao, The chodal norm of discrete mobius groups in several dimensions, Annal. Acad. Sci. Fenn. Math. 21(1996).
- [GM1] F.W. Gehring and G.J. Martin, Stability and extremality in Jørgensen's inequality, Complex Variables 12(1989).
- [GM2] F.W. Gehring and G.J. Martin, Discreteness in Kleinian groups and iteration theory (to appear)
- [GM3] F.W. Gehring and G.J. Martin, Inequalities for Mobius transformations and discrete groups, J. reine angew. Math. 418(1991)
- [M1] B. Maskit, *Kleinian groups*, Springer Verlag 1988.
- [M2] B. Maskit, Some special 2-generator Kleinian groups, Proc. Amer. Math. Soc. 106(1989).